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The electron energy spectrum of a circular ring in an external magnetic field

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Abstract. The dynamics of two-dimensional electrons in a circular ring in an external perpendicular magnetic field is investigated. The boundaries of the ring are approximated by infinite potential barriers. Within the quasiclassical approximation, a complete classification of the electron states generated by confinement and magnetic field is provided.

1. Introduction

Over the past years, technological progress in producing semiconductor devices by introducing lateral structure on quasi-two-dimensional electrons (see e.g. Merkt 1990, Merkt *et al* 1989, Kern *et al* 1991) has induced a vivid interest in theoretical investigations of confined electron dynamics (Geerinckx *et al* 1990, Rössler 1990, Lent 1991, Klama and Rössler 1992a, b, Falkovsky and Klama 1993). One of the frequently investigated models is an electron gas with the boundary simulated by a parabolic potential, i.e. a soft-wall potential (see e.g. Rössler 1990). This model eventually leads to simple analytical results; however, it also presents some shortcomings: for instance, it admits no edge states. In addition to the soft-wall potential model, another one with the two-dimensional (2D) region bounded by an infinite potential barrier (hard-wall potential) is studied. As there appears to be no possibility to get exact analytical results for the latter model with an external magnetic field, one applies numerical (Robnik 1986, Lent 1991) methods or the quasiclassical approach (Klama and Rössler 1992a, Falkovsky and Klama 1993). Within the quasiclassical approach, the rules leading to the electron energy spectrum (EES) represent transcendental equations—see Klama and Rössler (1992) and Falkovsky and Klama (1993) for the quantum dot and equations (31), (45) and (52) below for the ring—admitting analytical solutions in special cases only.

Nevertheless, the quasiclassical approximation makes it possible to check the character of the electron states generated by confinement and magnetic field, and to present their complete classification.

In this paper we study the energy spectrum of non-interacting electrons confined to a circular ring with hard-wall potentials in a perpendicular magnetic field.

2. The electron dynamics in 2D space in a magnetic field: quantum-mechanical approach

The problem of 2D electrons confined to a ring and subject to homogeneous magnetic field $\boldsymbol{H} = (0, 0, H)$ perpendicular to the ring can be modelled in the simple effective-mass

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approximation by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + V(x, y) \tag{1}$$

where

$$\mathcal{H}_0 = \frac{1}{2\mu}\left(p + \frac{e}{c}A\right)^2 \qquad p = (p_x, p_y, 0) \tag{2}$$

and $V(x, y)$ is the confinement potential. We choose the vector potential of the magnetic field in the symmetric gauge $A = \frac{1}{2}H(-y, x, 0)$ and introduce polar coordinates $(\rho, \varphi) = \rho$. For an infinite barrier confinement we consider the Schrödinger equation associated with $\mathcal{H}_0$ for the homogeneous 2D space with the boundary conditions of vanishing wavefunction at

$$\rho = r + 0 \qquad \rho = R - 0 \tag{3}$$

where r is the internal and R is the external radius of the ring. In the polar coordinate system the Hamiltonian (2) takes the form

$$\mathcal{H}_0(\rho) = -\frac{\hbar^2}{2\mu}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right] + \left[\frac{1}{2}\left(\frac{\rho}{l}\right)^2 - 2\left(\frac{l}{\rho}\right)^2\frac{\partial^2}{\partial\varphi^2} - 2\mathrm{i}\frac{\partial}{\partial\varphi}\right]\frac{\hbar\omega_c}{4} \tag{4}$$

where $l = (\hbar/\mu\omega_c)^{1/2}$ is the magnetic length and $\omega_c = eH/\mu c$. The angular momentum operator commutes with the Hamiltonian (4), hence we can take the wavefunction of the problem without confinement in the form

$$\Psi(\rho) = \Phi(\rho)\mathrm{e}^{\mathrm{i}m\varphi} \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{5}$$

Using (5) in the Schrödinger equation $\mathcal{H}(\rho)\Psi(\rho) = \epsilon\Psi(\rho)$ and making use of the *ansatz*

$$\Phi(\xi) = \xi^{m/2}\mathrm{e}^{-\xi/2}y(\xi) \qquad \xi = \tfrac{1}{2}(\rho/l)^2 \tag{6}$$

we obtain Kummer's differential equation:

$$\xi\,\mathrm{d}^2y(\xi)/\mathrm{d}\xi^2 + (\gamma - \xi)\,\mathrm{d}y(\xi)/\mathrm{d}\xi - \alpha y(\xi) = 0 \tag{7}$$

with $\gamma = 1 + m$, $\alpha = \gamma/2 - \lambda$ and $\lambda = (\epsilon/\hbar\omega_c) - m/2$.

The general solution of (7), for integer m, can be written as a linear combination of its linearly independent particular solutions $y_1(\xi)$ and $y_2(\xi)$ expressed by Kummer's functions F and U:

$$y(\xi) = C_1 y_1(\xi) + C_2 y_2(\xi) \tag{8}$$

where

$$y_1(\xi) = \xi^{(\bar{\gamma}-\gamma)/2}F(\bar{\alpha}, \bar{\gamma}, \xi) \qquad y_2(\xi) = U(\alpha, \gamma, \xi) \tag{9}$$

$\bar{\gamma} = 1 + |m|$, $\bar{\alpha} = \bar{\gamma}/2 - \lambda$ and C_1, C_2 are constants. Substituting (8) into (6) we get the general solution of the considered problem:

$$\Phi(\xi) = C_1\Phi_1(\xi) + C_2\Phi_2(\xi) \tag{10}$$

with $\Phi_i(\xi) = \xi^{m/2} e^{-\xi/2} y_i(\xi)$, $i = 1, 2$

From the asymptotic properties of the $\Phi_i(\xi)$ it follows that in the case of a free 2D space the solution (10) is bounded if we put $C_2 = 0$ and $\bar{\alpha} = -N$; $N = 0, 1, 2, \ldots$. These conditions lead us to the well known expression for the electron energy spectrum (EES):

$$\epsilon = \epsilon_N(m) = [N + \tfrac{1}{2}(|m| + m + 1)]\hbar\omega_c. \tag{11}$$

By imposing the boundary conditions (3) on the wavefunction (10), we can write the exact expression describing the EES of the ring:

$$\begin{vmatrix} \Phi_1(\xi_r) & \Phi_2(\xi_r) \\ \Phi_1(\xi_R) & \Phi_2(\xi_R) \end{vmatrix} = 0 \tag{12}$$

where $\xi_r = \frac{1}{2}(r/l)^2$ and $\xi_R = \frac{1}{2}(R/l)^2$. Equation (12) can be solved analytically in the cases of well defined asymptotic expressions for the functions F and U (see e.g. Abramowitz and Stegun 1964).

As an example we shall calculate the EES for the case of large $\bar{\alpha}$ and α and negative quantities and $\bar{\gamma}$ and γ that are bounded. In this case by using asymptotic representations for the functions F and U (see Abramowitz and Stegun 1964, formulae (13.5.14) and (13.5.16)) we obtain the following expression for the EES:

$$\epsilon = (N^2\omega_\Delta/(4\omega_c) + m/2)\hbar\omega_c \tag{13}$$

where $\hbar\omega_\Delta = (1/2\mu)[2\pi\hbar/(R-r)]^2$ is the energy of the electron with wavelength $R - r$. Equation (13) is valid under the condition $2\epsilon/\hbar\omega_c \gg |m| + m + 1$, i.e. for weak magnetic field.

3. The electron dynamics in 2D space: quasiclassical approach

For further discussion it is convenient to apply the quasiclassical approximation enabling us to arrive at expressions providing the electron wavefunctions in the whole range of values of parameters of the problem in question.

After substitution

$$y(\xi) = \xi^{-\gamma/2} e^{\xi/2} u(\xi) \tag{14}$$

into (7) and with $\kappa = \xi/2\lambda$ as a new variable, we obtain

$$\mathrm{d}^2u(\kappa)/\mathrm{d}\kappa^2 - \lambda^2 Q(\kappa)u(\kappa) = 0 \tag{15}$$

$$Q(\kappa) = 1 - 2/\kappa + \kappa_0/\kappa^2 \tag{16}$$

$$\kappa_0 = C/\lambda^2 = 4C(l/r_c)^4 \qquad \kappa = \tfrac{1}{2}(\rho/r_c)^2$$

$$C = (m^2 - 1)/4 \qquad r_c = l(2\lambda)^{1/2}. \tag{17}$$

Note that κ_0 determines the minimum of a 'potential' well described by $Q(\kappa)$ and $Q(\kappa_0) = 1 - 1/\kappa_0$ represents the depth of this well. The condition that bound states exist in the potential well is of the form $Q(\kappa_0) < 0$, and consequently $0 \leqslant \kappa_0 < 1$.

Now we shall solve (15) by the quasiclassical method. Since $Q(\kappa) \propto \kappa^{-2}$ at $\kappa \to 0$, the *correct solutions of the quasiclassical approximation can be obtained if in the expression for* C the term $(-1/4)$ is omitted; i.e. $C = (m/2)^2$ for $m \neq 0$ (see e.g. Nikiforov and Uvarov 1984). The case $m = 0$ has been discussed in detail by Falkovsky and Klama (1993).

Using (14) and (6) we get

$$\Phi(\xi) = \xi^{-1/2}u(\xi) = \xi^{-1/2}[C_1 u_1(\xi) + C_2 u_2(\xi)] \tag{18}$$

where u_1 and u_2 are the particular solutions of (15). Making use of the well known rules for matching wavefunctions at the turning points $\kappa_1 = 1 - (1-\kappa_0)^{1/2}$ and $\kappa_2 = 1 + (1-\kappa_0)^{1/2}$ of the 'potential' well $Q(\kappa)$, we can write the following expressions for the solutions of (15) (cf. Falkovsky and Klama 1993):

$$u_1(\kappa) = \begin{cases} \dfrac{B}{Q^{1/4}(\kappa)} \exp\left(-\lambda \displaystyle\int_\kappa^{\kappa_1} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) & \text{for } 0 < \kappa < \kappa_1 \quad (19a) \\ \dfrac{2B}{|Q(\kappa)|^{1/4}} \cos\left(\lambda \displaystyle\int_{\kappa_1}^{\kappa} \mathrm{d}\kappa [|Q(\kappa)|]^{1/2} - \pi/4\right) & \text{for } \kappa_1 < \kappa < \kappa_2 \quad (19b) \\ \dfrac{B}{Q^{1/4}(\kappa)}\Bigg[\exp\left(-\lambda \displaystyle\int_{\kappa_2}^{\kappa} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) \sin\left(\lambda \displaystyle\int_{\kappa_1}^{\kappa_2} \mathrm{d}\kappa [|Q(x)|]^{1/2}\right) & \\ \quad + 2\exp\left(\lambda \displaystyle\int_{\kappa_2}^{\kappa} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) \cos\left(\lambda \displaystyle\int_{\kappa_1}^{\kappa_2} \mathrm{d}\kappa [|Q(\kappa)|]^{1/2}\right)\Bigg] & \\ & \text{for } \kappa > \kappa_2 \quad (19c) \end{cases}$$

$$u_2(\kappa) = \begin{cases} \dfrac{A}{Q^{1/4}(\kappa)} \exp\left(-\lambda \displaystyle\int_{\kappa_2}^{\kappa} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) & \text{for } \kappa > \kappa_2 \quad (20a) \\ \dfrac{2A}{|Q(\kappa)|^{1/4}} \cos\left(\lambda \displaystyle\int_{\kappa_2}^{\kappa} \mathrm{d}\kappa [|Q(\kappa)|]^{1/2} + \pi/4\right) & \text{for } \kappa_1 < \kappa < \kappa \quad (20b) \\ \dfrac{A}{Q^{1/4}(\kappa)}\Bigg[2\exp\left(\lambda \displaystyle\int_{\kappa}^{\kappa_1} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) \cos\left(\lambda \displaystyle\int_{\kappa_1}^{\kappa_2} \mathrm{d}\kappa [|Q(\kappa)|]^{1/2}\right) & \\ \quad + \exp\left(-\lambda \displaystyle\int_{\kappa}^{\kappa_1} \mathrm{d}\kappa [Q(\kappa)]^{1/2}\right) \sin\left(\lambda \displaystyle\int_{\kappa_1}^{\kappa_2} \mathrm{d}\kappa [|Q(\kappa)|]^{1/2}\right)\Bigg] & \\ & \text{for } 0 < \kappa < \kappa_1 \quad (20c) \end{cases}$$

with

$$\begin{aligned} A &= \exp[-\lambda(1 - \ln\lambda)] \\ B &= (|m|!/2\pi)\Gamma(1-\bar{\alpha})\exp[\lambda(1 - \ln\lambda)] \end{aligned} \tag{21}$$

where $\Gamma(x)$ is the Euler Γ function. In the expressions (19c) and (20c) there exist not only large terms but also small ones, which become relevant if the coefficients of the large terms disappear. In such a case we have substituted in the small terms $(-1)^N$ for the sine functions (Falkovsky and Klama 1993). The expressions (19) and (20) hold under the condition

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\{\kappa[(\kappa - \kappa_1)(\kappa_2 - \kappa)]^{-1/2}\} \ll \lambda$$

which represents the condition for the quasiclassical approximation to be valid. Hence, the results obtained using the wavefunctions (19) and (20) are formally valid at large N and arbitrary $|m|$, or inversely.

The expression (18) with u_1 and u_2 in the form (19) and (20) represent the quasiclassical solution of the considered problem. From (9) and (14) we get expressions:

$$F(\bar{\alpha}, \bar{\gamma}, \xi) = \xi^{-\bar{\gamma}/2} e^{\xi/2} u_1(\xi) \tag{22a}$$

$$U(\alpha, \gamma, \xi) = \xi^{-\gamma/2} e^{\xi/2} u_2(\xi) \tag{22b}$$

which together with (19) and (20) are the quasiclassical representations of Kummer's functions.

After imposing the boundary conditions (3) on the quasiclassical wavefunction (18) we obtain the expression determining the EES of the ring in the quasiclassical approximation:

$$\begin{vmatrix} u_1(\kappa_r) & u_2(\kappa_r) \\ u_1(\kappa_R) & u_2(\kappa_R) \end{vmatrix} = 0 \tag{23}$$

with $\kappa_r = \frac{1}{2}(r/r_c)^2$ and $\kappa_R = \frac{1}{2}(R/r_c)^2$.

Within the quasiclassical approach the electron states are represented by cyclotron orbits. Hence the interaction of an electron with the boundaries of a finite region reduces to reflection of an electron moving on a circular cyclotron orbit at the boundaries in such a way that only a fragment of that cyclotron orbit which would exist in an unbounded 2D space can subsist within the ring. Such fragments form a cyclotron trajectory lying within the ring. The character of this trajectory depends on mutual relations among the values of the parameters of the cyclotron orbit in the unbounded 2D space, σ and γ, and the linear dimensions of the ring, r and R. The symbol σ represents the cyclotron radius and γ is the cyclotron orbit guide centre. Note that the quantities γ and σ are the constants of motion in both the classical and the quantum-mechanical sense in the unbounded 2D space and fulfil the relations:

$$\gamma^2 + \sigma^2 = 2r_c^2 \qquad \gamma^2 - \sigma^2 = -2ml^2. \tag{24}$$

For the purpose of further discussion we shall present relations between the characteristics of the cyclotron orbits and the variables characterizing the EES of the ring.

Note that the expressions (24) are of approximate character as far as the problem of objects bounded in space is considered (cf. Lent 1991). Nevertheless, we shall apply this approximation with the aim of classifying the cyclotron trajectories with respect to the values of the parameters σ, γ and r, R. Let us stress that in the model under consideration the reflection of the electron from the boundaries is of a specular character. After making use of (24) and (17) we get

$$\begin{aligned} &\kappa_0^{1/2} = \begin{cases} (\gamma^2 - \sigma^2)/(\gamma^2 + \sigma^2) & \text{for } m < 0 \\ (\sigma^2 - \gamma^2)/(\gamma^2 + \sigma^2) & \text{for } m > 0 \end{cases} \\ &\kappa_1 = (\gamma - \sigma)^2/(\gamma^2 + \sigma^2) \qquad \kappa_2 = (\gamma + \sigma)^2/(\gamma^2 + \sigma^2) \\ &\kappa_r = r^2/(\gamma^2 + \sigma^2) \qquad \kappa_R = R^2/(\gamma^2 + \sigma^2). \end{aligned} \tag{25}$$

From (24) it follows that $\gamma < \sigma$ for $m > 0$ and $\gamma > \sigma$ for $m < 0$.

Equation (23) and expressions (19), (20) and (24) allow us in the quasiclassical approximation to find expressions describing the EES and to classify the electron states according to the related cyclotron orbits. The expressions for the quasiclassical wavefunctions (see (19) and (20)) allow one to perform an analysis of the problem in question by considering four separate cases covering the whole range of values of the parameters of our problem. The cases are determined by mutual relations of κ_r, κ_R and κ_1, κ_2 as follows:

$$(\mathcal{A})\ \kappa_1 < \kappa_r < \kappa_2 \qquad \text{and} \qquad \kappa_R < \kappa_2 \tag{26}$$

$$(\mathcal{B})\ \kappa_1 < \kappa_r < \kappa_2 \qquad \text{and} \qquad \kappa_R > \kappa_2 \tag{27}$$

$$(\mathcal{C})\ 0 < \kappa_r < \kappa_1 \qquad \text{and} \qquad \kappa_1 < \kappa_R < \kappa_2 \tag{28}$$

$$(\mathcal{D})\ 0 < \kappa_r < \kappa_1 \qquad \text{and} \qquad \kappa_R > \kappa_2. \tag{29}$$

For all those cases $\kappa_r < \kappa_R$, since for the ring $r < R$. Let us discuss the above cases one by one.

3.1. *The case* ($\mathcal{A}$)

In this case both potential barriers are situated between the turning points κ_1 and κ_2 (see (26)). Substituting the corresponding quasiclassical expressions for wavefunctions (19) and (20) into (23) we arrive at the expression describing the EES of the states that arise via interaction between the electron and both boundaries of the ring:

$$\lambda \int_{\kappa_r}^{\kappa_R} \mathrm{d}\kappa\, [|Q(\kappa)|]^{1/2} = \pi N. \tag{30}$$

After performing the integration in (30) we obtain the equation determining the energy spectrum of these states:

$$f(\kappa_R, \kappa_r, \kappa_0) = \Phi(N, |m|) \tag{31}$$

where

$$\begin{aligned} f(\kappa_r, \kappa_R, \kappa_0) = \frac{1}{2\pi\kappa_0^{1/2}} \Bigg\{ & (2\kappa_R - \kappa_R^2 - \kappa_0)^{1/2} - (2\kappa_r - \kappa_r^2 - \kappa_0)^{1/2} \\ & - \sin^{-1}\left(\frac{1-\kappa_R}{(1-\kappa_0)^{1/2}}\right) + \sin^{-1}\left(\frac{1-\kappa_r}{(1-\kappa_0)^{1/2}}\right) \\ & - \kappa_0^{1/2}\left[\sin^{-1}\left(\frac{1-\kappa_0/\kappa_R}{(1-\kappa_0)^{1/2}}\right) - \sin^{-1}\left(\frac{1-\kappa_0/\kappa_r}{(1-\kappa_0)^{1/2}}\right)\right]\Bigg\} \end{aligned} \tag{32}$$

with

$$\Phi(N, |m|) = N/|m| \qquad m = \pm 1, \pm 2, \pm 3, \ldots. \tag{33}$$

Now we introduce the scaling parameter

$$\nu = \kappa_r/\kappa_R = (r/R)^2 < 1 \tag{34}$$

and a new variable, which will characterize the region of motion accessible for the electron in the ring:

$$\bar{\kappa} = \tfrac{1}{8}[(R-r)/r_c]^2 = [(1-\nu^{1/2})^2/(4\nu)]\kappa_r. \tag{35}$$

From (34) and (35) we get

$$\kappa_r = \tau\bar{\kappa} \qquad \tau = \tau(\nu) = 4\nu/(1-\nu^{1/2})^2 \qquad \kappa_R = \tau\bar{\kappa}/\nu. \tag{36}$$

In these new variables equation (31) takes the form

$$f(\bar{\kappa}, \kappa_0; \nu) = \Phi(N, |m|) \tag{37}$$

with

$$\begin{aligned} f(\bar{\kappa}, \kappa_0; \nu) = \frac{1}{2\pi\kappa_0^{1/2}} \Bigg\{ & [2\tau\bar{\kappa}/\nu - (\tau\bar{\kappa}/\nu)^2 - \kappa_0]^{1/2} - (2\tau\bar{\kappa} - \tau^2\bar{\kappa}^2 - \kappa_0)^{1/2} \\ & - \sin^{-1}\left(\frac{1-\tau\bar{\kappa}/\nu}{(1-\kappa_0)^{1/2}}\right) + \sin^{-1}\left(\frac{1-\tau\bar{\kappa}}{(1-\kappa_0)^{1/2}}\right) \\ & - \kappa_0^{1/2}\left[\sin^{-1}\left(\frac{1-\nu\kappa_0/\tau\bar{\kappa}}{(1-\kappa_0)^{1/2}}\right) - \sin^{-1}\left(\frac{1-\kappa_0/\tau\bar{\kappa}}{(1-\kappa_0)^{1/2}}\right)\right]\Bigg\}. \end{aligned} \tag{38}$$

The region of physical solutions of (37) in the $(\bar{\kappa}, \kappa_0^{1/2})$ plane, called the K plane in the following, is defined by the following inequalities:

$$(1-\bar{\kappa}\tau/\nu)^2 + \kappa_0 < 1 \tag{39}$$

$$(1-\bar{\kappa}\tau)^2 + \kappa_0 < 1 \tag{40}$$

$$\bar{\kappa} > \nu\bar{\kappa} \qquad (\nu < 1). \tag{41}$$

The above-mentioned parameters fulfil the relations

$$\begin{gathered} 0 < \nu < 1 \qquad 0 < \tau < \infty \qquad \nu/\tau = (1-\nu^{1/2})^2/4 \\ 0 < \nu/\tau \leqslant 1/4 \qquad 1/\tau \in [0, \infty). \end{gathered}$$

Equations corresponding to the inequalities (39) and (40) as readily seen describe semiellipses in the K plane. To (39) correspond the semiellipse S (connected with R barrier) and to (40) the semiellipse L (connected with r barrier). On this plane the considered electron states are positioned inside the overlap region of these semiellipses (see figure 1). The upper boundary (upper part of the semiellipse S) of this region describes the equation $\bar{\kappa} = \nu\kappa_2/\tau$ and the lower one (lower part of the semiellipse L) $\bar{\kappa} = \kappa_1/\tau$. The upper part of the semiellipse L is described by the equation $\bar{\kappa} = \kappa_2/\tau$, and the lower part of the semiellipse S is described by the equation $\bar{\kappa} = \nu\kappa_1/\tau$. In the case $1/\tau < 1$ $(1 < \tau < \infty)$ and $1/9 < \nu < 1$ (narrow ring) both semiellipses are elongated along the $\kappa_0^{1/2}$ axis (see figures 1(*a*) and (*c*)), but if $1/\tau > 1$ $(0 \leqslant \tau < 1)$ and $0 \leqslant \nu \leqslant 1/9$ (wide ring), the semiellipse L is elongated along the $\bar{\kappa}$ axis and the semiellipse S is elongated along the $\kappa_0^{1/2}$ axis (see figures 1(*b*) and (*d*)).

The electronic states are represented by three topologically different cyclotron trajectories that arise due to the specular reflection of the electron moving along the cyclotron orbit on both boundaries of the ring. If the centre of the cyclotron orbit is situated within the circle of radius r, the parameters of the problem under consideration fulfil the following relations:

$$0 < r - \gamma < \sigma \qquad R - \gamma < \sigma \qquad \text{i.e.} \qquad \kappa_3 < \kappa_r < \kappa_2 \qquad \kappa_3 < \kappa_R < \kappa_2 \tag{42}$$

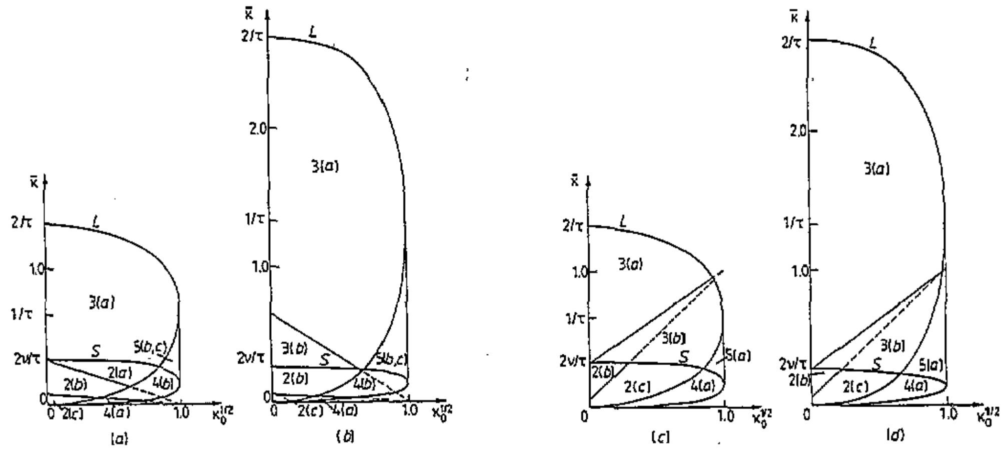

Figure 1. Semiellipses in the $(\bar{\kappa}, \kappa_0^{1/2})$ plane (explained in the text) indicating the regions for which the solutions of (31), (45) and (52) represent the electron states corresponding to topologically different cyclotron trajectories within the ring in the presence of an external magnetic field: (*a*) and (*b*) for $m < 0$, (*c*) and (*d*) for $m > 0$; (*a*) and (*c*) for the case $1/\tau < 1$ and $1/9 < \nu < 1$ $(1/3 < r/R < 1)$, (*b*) and (*d*) for the case $1/\tau > 1$, $0 \leqslant \nu \leqslant 1/9$ $(0 < r/R < 1/3)$. Parts (*a*) and (*b*) were drawn for $\nu = 0.25$ and (*c*) and (*d*) were drawn for $\nu = 0.1$. The regions where various types of cyclotron trajectories exist (explanation in the text) are separated by full lines and arcs. In the regions denoted by 2(*a*), 2(*b*), 2(*c*), 3(*a*), 3(*b*), 3(*c*), 4(*a*), 4(*b*), 5(*a*), 5(*b*) and 5(*c*) there exist the types of cyclotron trajectories shown in figures 2, 3, 4 and 5, respectively.

where

$$\kappa_3 = \gamma^2/(\gamma^2 + \sigma^2). \tag{42a}$$

This kind of cyclotron trajectory is shown on figure 2(*a*). The quantity κ_3 can be expressed with the help of (17) and (24) as

$$\kappa_3 = \tfrac{1}{2}(1 - m/2\lambda) = \tfrac{1}{2}(1 \pm \kappa_0^{1/2}) \tag{42b}$$

where the + sign stands for $m < 0$ and − sign for $m > 0$. On the K plane the cyclotron trajectories (CT) of 2(*a*) type exist in the region $\kappa_3/\tau < \bar{\kappa} < \nu\kappa_2/\tau$. This region is bounded from the right by the arc $\bar{\kappa} > \kappa_1/\tau$.

If the centre of the cyclotron orbit is situated within the ring $(r < \gamma < R)$ the trajectory shown on figure 2(*b*) is realized and the parameters fulfil the following inequalities

$$0 < \gamma - r < \sigma \qquad R - \gamma < \sigma \qquad \text{i.e.} \qquad \kappa_1 < \kappa_r < \kappa_3 \qquad \kappa_3 < \kappa_R < \kappa_2. \tag{43}$$

It follows that CT of this type exist at

$$\nu\kappa_3/\tau < \bar{\kappa} < \kappa_3/\tau \tag{43a}$$

and if the condition of existence of such CT within the region of semiellipses overlap is added to (43*a*), i.e. $\bar{\kappa} > \kappa_1/\tau$, we shall obtain a complete description of the region in the K plane where these CT exist (see figure 1).

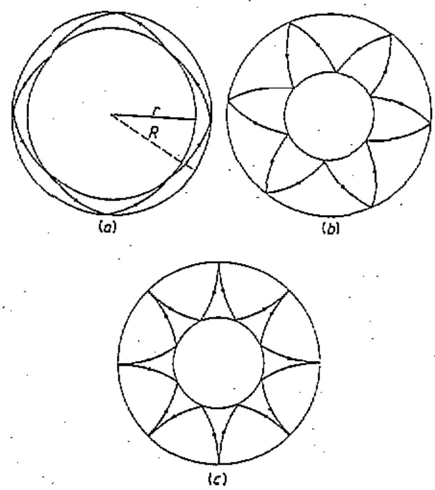

Figure 2. The types of cyclotron trajectories created by specular reflection of an electron moving along cyclotron orbits with reflection occurring on both boundaries of the ring. The EES related to these trajectories can be obtained as a solution of (31). Localization of these states in the $(\bar{\kappa}, \kappa_0^{1/2})$ plane is shown in figure 1 and conditions of their existence are given in figure 6.

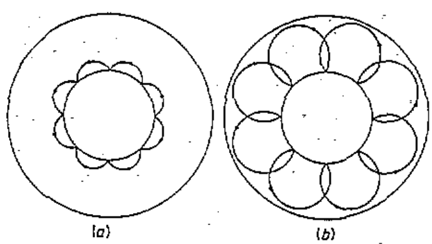

Figure 3. The types of cyclotron trajectories created by specular reflection of an electron moving along cyclotron orbits with reflection occurring along the internal boundary of the ring. These trajectories lie in the regions of the $(\bar{\kappa}, \kappa_0^{1/2})$ plane that are shown in figure 1 and exist under conditions given in figure 6. The EES related to these trajectories is described by equation (45).

If the centre of the cyclotron orbit is situated outside the ring ($\gamma > R$) we have:

$$0 \leqslant \gamma - R < \sigma \qquad \gamma - r < \sigma \qquad \text{i.e.} \qquad \kappa_1 < \kappa_r < \kappa_3 \qquad \kappa_1 < \kappa_R < \kappa_3. \tag{44}$$

This kind of trajectory is illustrated on figure 2(c). The CT of 2(c) type exist on the K plane at $\kappa_1/\tau < \bar{\kappa} < \nu\kappa_3/\tau$ (see figure 1).

For $\kappa_0 \ll \kappa_R \ll 1$, $\kappa_r < \kappa_R$ in the RHS of the expression (38) we can apply the expansion:

$$\sin^{-1} x \simeq \pi/2 - (1 - x^2)^{1/2}/x$$

and from (31) we obtain the expression (13) for the EES in the weak magnetic field region.

3.2. *The case* ($\mathcal{B}$)

As follows from (27), now the lowest potential barrier (determined by κ_r) is confined between the turning points, whereas the upper barrier lies outside the turning point κ_2. Here, the electron is specularly reflected by the internal boundary of the ring, i.e. $\rho = r$.

Substituting the corresponding quasiclassical wavefunctions (19) and (20) into (23) we arrive at the expression describing the electron energetic spectrum of the edge states (skipping orbits on the boundary $\rho = r$). These states arise due to interaction between the electron and the internal boundary of the ring. The expression has the form

$$\cos\alpha - 2\,\mathrm{e}^{2J_R(\lambda)}\cos[\lambda\pi(1-\kappa_0^{1/2})]\cos\beta = 0 \tag{45}$$

where

$$\alpha = \lambda\int_{\kappa_1}^{\kappa_r} \mathrm{d}\kappa\,[|Q(\kappa)|]^{1/2} - \frac{\pi}{4} \qquad \beta = \lambda\int_{\kappa_r}^{\kappa_2} \mathrm{d}\kappa\,[|Q(\kappa)|]^{1/2} - \frac{\pi}{4} \tag{46}$$

$$J_R(\lambda) = \lambda\int_{\kappa_2}^{\kappa_R} \mathrm{d}\kappa\,[Q(\kappa)]^{1/2}. \tag{47}$$

In this case we have two types of cyclotron trajectories, which are shown in figure 3. If the cyclotron orbit centre lies inside the circle with radius r ($\gamma < r$), the parameters of the considered system fulfil the relations:

$$0 < r - \gamma < \sigma \qquad R - \gamma > \sigma \qquad \text{i.e.} \qquad \kappa_3 < \kappa_r < \kappa_2 \qquad \kappa_R > \kappa_2. \tag{48}$$

This type of CT is shown in figure 3(a). The CT of 3(a) type consist of fragments of cyclotron orbits traced by the moving electron specularly reflected along the r barrier (with the cyclotron orbit centre inside of the circle with radius r); these are skipping trajectories on the r barrier of the ring. In the K plane these CT exist at $\nu\kappa_2/\tau < \bar{\kappa} < \kappa_2/\tau$ and $\bar{\kappa} > \kappa_3/\tau$. We note that this type of CT exists for an integer m, but the area of this region occupied by them on the K plane depends on the sign of m and the values of the parameters τ and ν (see figure 1).

In the second case when the centre of the cyclotron orbit lies inside of the ring ($r < \gamma < R$) we have:

$$0 < \gamma - r < \sigma \qquad R - \gamma > \sigma \qquad \text{i.e.} \qquad \kappa_1 < \kappa_r < \kappa_3 \qquad \kappa_R > \kappa_2. \tag{49}$$

The cyclotron trajectory is shown in figure 3(b). The CT of 3(b) type consist of parts of cyclotron orbits traced by the moving electron specularly reflected along the r barrier. They occur if $\nu\kappa_2/\tau < \bar{\kappa} < \kappa_3/\tau$ and $\kappa_3/\tau > \nu\kappa_2/\tau$, and exist in the regions of the K plane shown in figure 1.

3.3. *The case* (*C*)

In this case (see (28)) the electron is specularly reflected by the boundary $\rho = R$. The lower potential barrier lies below the left turning point and the upper barrier lies outside the right turning point. The presence of the specular internal boundary of the ring affects only the exponentially damped tail of the wavefunction. Here also two types of trajectories exist: namely such with the cyclotron radius centre lying either outside (figure 4(a)) or inside the ring (figure 4(b)).

If the cyclotron orbit centre lies outside the ring (see figure 4(a)) then

$$\gamma - r > \sigma \qquad 0 < \gamma - R < \sigma \qquad \text{i.e.} \qquad 0 < \kappa_r < \kappa_1 \qquad \kappa_1 < \kappa_R < \kappa_3. \tag{50}$$

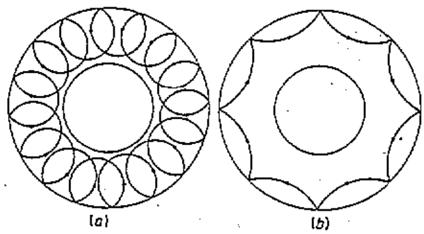

Figure 4. The types of cyclotron trajectories created by specular reflection of an electron moving along cyclotron orbits with reflection occurring along the external boundary of the ring. These trajectories lie in the regions of the $(\bar{\kappa}, \kappa_0^{1/2})$ plane shown in figure 1 and conditions for their existence are given in figure 6. The EES related ot these trajectories is described by equation (52).

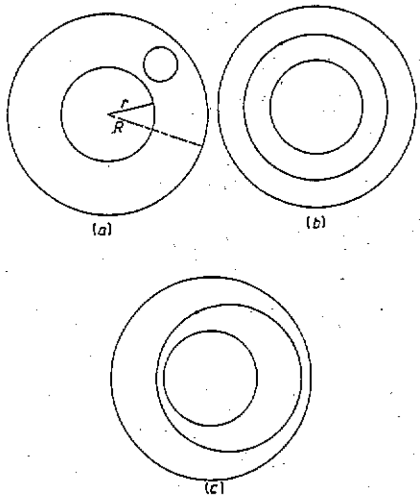

Figure 5. The types of cyclotron orbits with the electron interacting weakly with the boundaries of the ring. They are situated in the region of the $(\bar{\kappa}, \kappa_0^{1/2})$ plane shown in figure 1 and exist under conditions given in figure 6. Type (a) exists for $m < 0$ and types (b) and (c) for $m > 0$. The EES corresponding to these electron states is described by equation (55).

The CT of 4(a) type represent skipping trajectories along the R barrier with the cyclotron orbit centre outside of the ring. The region of existence of this kind of the CT in the K plane is described by the conditions: $\nu\kappa_1/\tau < \bar{\kappa} < \kappa_1/\tau$ and $\kappa_1/\tau < \nu\kappa_3/\tau$.

In the case when the cyclotron orbit centre lies inside the ring (see figure 4(b)) we have:

$$\gamma - r \geqslant \sigma \qquad 0 < R - \gamma < \sigma \qquad \text{i.e.} \qquad 0 < \kappa_r \leqslant \kappa_1 \qquad \kappa_3 < \kappa_R < \kappa_2. \tag{51}$$

The CT of 4(b) type represent skipping trajectories along the R barrier. These CT consist of fragments of cyclotron orbits of the electron reflected specularly from the R boundary of the ring and these orbits have their centres inside the ring. The existence conditions for these CT on the K plane are as follows: $\nu\kappa_3/\tau < \bar{\kappa} < \kappa_2/\tau$. The CT of 4$(a)$ and 4(b) type lie in that part of the semiellipse S which has no overlap with the semiellipse L (see figure 1); however, the electron states of 4(b) type do not exist for $m > 0$.

The expression describing the spectrum of the states of 4(a) and 4(b) type can be obtained by substitution of the corresponding wavefunctions (19) and (20) into (23). We obtain

$$\cos\beta_R - 2\,\mathrm{e}^{2J_r(\lambda)}\cos[\lambda\pi(1-\kappa_0^{1/2})]\cos\alpha_R = 0 \tag{52}$$

where

$$\beta_R = \lambda\int_{\kappa_R}^{\kappa_2} \mathrm{d}\kappa\,[|Q(\kappa)|]^{1/2} - \frac{\pi}{4} \qquad \alpha_R = \lambda\int_{\kappa_1}^{\kappa_R} \mathrm{d}\kappa\,[|Q(\kappa)|]^{1/2} - \frac{\pi}{4} \tag{53}$$

$$J_r(\lambda) = \lambda\int_{\kappa_r}^{\kappa_1} \mathrm{d}\kappa\,[Q(\kappa)]^{1/2}. \tag{54}$$

3.4. The case ($\mathcal{D}$)

Now, both potential barriers lie outside the turning points, i.e. $0 < \kappa_r < \kappa_1$ and $\kappa_R > \kappa_2$. In this case the EES of the ring has Landau character. After inserting the corresponding wavefunctions (19) and (20) into (23) we obtain the expression describing the EES of these states in the following form:

$$\epsilon = \epsilon_{N,m} = [N + \tfrac{1}{2}(|m| + m) + 1]\hbar\omega_c - (\hbar\omega_c/\pi)\cos^{-1}[\tfrac{1}{2}(\mathrm{e}^{-2J_R(\lambda)} + \mathrm{e}^{-2J_r(\lambda)})] \tag{55}$$

$$\approx [N + \tfrac{1}{2}(|m| + m + 1)]\hbar\omega_c + (\hbar\omega_c/2\pi)(\mathrm{e}^{-2J_R(\lambda)} + \mathrm{e}^{-2J_r(\lambda)}) \tag{55a}$$

where

$$J_r(\lambda) = \lambda\int_{\kappa_r}^{\kappa_1} \mathrm{d}\kappa\,[Q(\kappa)]^{1/2} \qquad J_R(\lambda) = \lambda\int_{\kappa_2}^{\kappa_R} \mathrm{d}\kappa\,[Q(\kappa)]^{1/2}. \tag{56}$$

The integrals (56) should be calculated at a value of the variable ϵ determined by the first term on the RHS of expression (55a).

The cyclotron orbits representing these states are shown in figure 5. The parameters of the cyclotron orbits shown on figure 5(a) fulfil the inequalities $\gamma - r > \sigma$ and $R - \gamma > \sigma$ and the corresponding electron states exist for $m < 0$. For the cyclotron orbits symmetric with respect to the coordinate system centre shown in figure 5(b) it occurs at $\gamma = 0$, $r < \sigma < R$ and the electron states exist for $m > 0$. The cyclotron orbits shown in figure 5(c) exist for $m > 0$ at

$$R - \gamma > \sigma \qquad \gamma > r \qquad \gamma + r < \sigma \qquad m > 0. \tag{57}$$

The CT of 5(a), 5(b) and 5(c) type are of Landau character. As mentioned before, these states are situated in the region outside the semiellipses S and L. The region where the cyclotron orbits of 5(a), 5(b) and 5(c) type exist on the K plane is described by the following inequalities: $\nu\kappa_2/\tau < \bar{\kappa} < \kappa_1/\tau$. The right boundary of this region is described by the equation $\kappa_0 = 1$.

A scheme of conditions for various kinds of the cyclotron trajectories in the ring is given in figure 6.

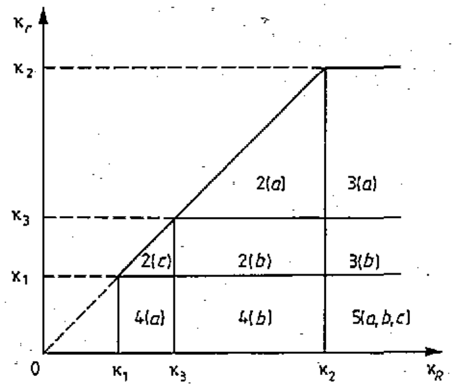

Figure 6. The existence regions of various topologically different CT as shown in figures 2–5 presented in terms of variables κ_r and κ_R. The κ_1 and κ_2 represent the turning points whereas the positions of the half-lines $\kappa_r = \kappa_3$ and $\kappa_R = \kappa_3$ depend on the sign of m (see (42*b*)).

4. Summary

Application of the quasiclassical approximation to the problem of the energy spectrum of an electron confined to a ring made it possible:

(i) to perform a complete classification of the electron states in the ring in a perpendicular magnetic in the whole range of values of parameters of the problem in question,

(ii) to find the existence conditions for particular electronic states (cf figure 6),

(iii) to write analytical expressions determining the EES for low (cf (13), see also the text after (44)) and high magnetic fields (cf (55)), and

(iv) to derive simple transcendental equations determining the EES for any values of the parameters of the system (cf (31), (45) and (52)).

Equations (31), (45) and (52) are easy to solve numerically in the K plane and can be rewritten with no difficulty in the commonly used coordinates of energy versus magnetic field (cf Klama and Rössler 1992a). The same applies to the regions of existence of particular types of electronic states shown in figure 6. Note that the transcendental equations as well as the existence conditions of the electronic states bear a considerably more complicated form when expressed in the energy and magnetic field variables than that in the κ and κ_0 ones.

After completing this work there appeared the interesting paper by Chakraborty and Pietiläinen (1992) concerning quantum dots and rings with parabolic-wall potential, and pointing out that in semiconductor nanostructures a quantum ring can be created from a quantum dot. The perspective of publication of experimental papers dealing with the ring problem in the nearest future makes it reasonable to perform a numerical analysis of the above-mentioned transcendental equations for specific physical systems.

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